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コンパクト次数 cmp に関する de Groot と Nishiura の問題

Vitalij A. Chatyrko (Linköping University)
服部泰直 (島根大学総合理工学部)

1 Introduction

A regular space X is called *rim-compact* if there exists a base $\mathcal{B}$ for the open sets of X such that the boundary Bd U is compact for each U in $\mathcal{B}$.

In 1942 de Groot (cf. [1]) proved the following:
(*) *A separable metrizable space X is rim-compact if and only if there is a metrizable compactification Y of X such that* ind $(Y \setminus X) \leq 0$.

In an attempt to generalize (*), de Groot introduced two notions, *the small inductive compactness degree cmp* and *the compactness definiency def* (we will recall the definitions in Section 2 and Section 3 respectively). It is known that the inequality cmp $X \leq$ def X holds for every separable metrizable space X. The well known conjecture of de Groot (see for example [4]) was that the two invariants coincide in the class of separable metrizable spaces. As a way either to disprove or to support the conjecture de Groot and Nishiura [4] posed the following:

Question 1.1 *Let $Z_n = [0,1]^{n+1} \setminus (0,1)^n \times \{0\}$. Is it true that cmp$Z_n \geq n$ for $n \geq 3$?*

In the quoted article, de Groot and Nishiura proved that def $Z_n = n$ for every $n \geq 1$, and they also stated that cmp $Z_i = i$ for $i = 1, 2$.

In [9], R. Pol constructed a space $P \subset R^4$ such that cmp $P = 1 <$ def $P = 2$. The space P is a modification of an example given by Luxemburg [7] of a compactum with noncoinciding transfinite inductive dimensions. After that, some other counterexamples to the de Groot's conjecture were constructed by Hart (cf. [1]), Kimura [6], Levin and Segal [8]). However, Question 1.1 remained open (see also [10, Question 418] and [1, Problem 3, page 71]).

One of our main results is the following.

Theorem 1.1 *Let $n \leq 2^m - 1$ for some integer m. Then cmp $Z_n \leq m+1$. In particular cmp $Z_n <$ def Z_n for $n \geq 5$.*

This is the answer to Question 1.1 for $n \geq 5$. Our paper is based on a construction of examples of compacta with noncoinciding transfinite inductive dimensions given in [2]. Our terminology follows [5] and [1].

2 Finite sum theorem for $\mathcal{P}$-ind

In this part, topological spaces are assumed to be regular T_1 and all classes of topological spaces considered are assumed to be nonempty and to contain any space homeomorphic with a closed subspace of one of their members. The letter $\mathcal{P}$ is used to denote such classes.

Recall the definition of *the small inductive dimension modulo $\mathcal{P}$, $\mathcal{P}$-ind* . Let X be a space.

(i) $\mathcal{P}\text{-ind } X = -1$ iff $X \in \mathcal{P}$;

(ii) $\mathcal{P}\text{-ind } X \leq n$ (≥ 0) if each point in X has arbitrarily small neighbourhoods V with $\mathcal{P}\text{-ind Bd } V \leq n-1$.

(iii) $\mathcal{P}\text{-ind } X = n$ if $\mathcal{P}\text{-ind } X \leq n$ and $\mathcal{P}\text{-ind } X > n-1$;

(iv) $\mathcal{P}\text{-ind } X = \infty$ if $\mathcal{P}\text{-ind } X > n$ for $n = -1, 0, 1, ...$

It is clear that if $\mathcal{P} = \{\emptyset\}$ then $\mathcal{P}\text{-ind } X = \text{ind } X$. If $\mathcal{P}$ is the class of compact spaces then $\mathcal{P}\text{-ind } X = \text{cmp } X$.

The following is a list of properties of $\mathcal{P}$-ind we shall use in the paper.

(1) If A is closed in X then $\mathcal{P}\text{-ind } A \leq \mathcal{P}\text{-ind } X$.

(2) If $\mathcal{P}\text{-ind } X \leq n \geq 0$ and U is open in X then $\mathcal{P}\text{-ind } U \leq n$.

(3) If $X = O_1 \cup O_2$, where O_i is open in $X, i = 1, 2$, and $\max\{\mathcal{P}\text{-ind } O_1, \mathcal{P}\text{-ind } O_2\} \leq n \geq 0$. Then $\mathcal{P}\text{-ind } X \leq n$.

(4) $\mathcal{P}\text{-ind } X \leq n \geq 0$ iff for each point p and for each closed set G of X with $p \notin G$ there is a partition S between p and G such that $\mathcal{P}\text{-ind } S \leq n-1$.

The following statement is contained implicitly in the proofs of [2, Theorem 3.9] and [3, Theorem 2].

Lemma 2.1 . *Let X be a normal space such that $X = X_1 \cup X_2$, where X_i is closed in X, and A, B be two closed disjoint subsets of X such that $A \cap X_i \neq \emptyset$ and $B \cap X_i \neq \emptyset, i = 1, 2$. Choose a partition C_1 in X_1 between the sets $A \cap X_1$ and $B \cap X_1$ such that $X_1 \setminus C_1 = U_1 \cup V_1$, where U_1, V_1 are open in X_1 and disjoint, and $A \cap X_1 \in U_1$, $B \cap X_1 \subset V_1$. Choose also a partition C_2 in X_2 between the the sets $A \cap X_2$ and $((C_1 \cup V_1) \cup B) \cap X_2$ such that $X_2 \setminus C_2 = U_2 \cup V_2$, where U_2, V_2 are open in X_2 and disjoint, and $A \cap X_2 \in U_2$, $(C_1 \cup V_1) \cup B) \cap X_2 \subset V_2$. Then the set $C = X \setminus (((U_1 \setminus X_2) \cup U_2) \cup (V_1 \cup (V_2 \setminus X_1)))$ is a partition in X between the sets A and B such that $C \subset C_1 \cup C_2 \cup (X_1 \cap X_2)$.*
Moreover, if X is a regular T_1-space then the same statement is valid for a pair of closed subsets of X, where one of the sets is a point.

The following theorem and corollary are generalizations of [3, Theorem 2] and [2, Corollary 3.10 (a)] respectively.

Theorem 2.1 *Let X be a space such that $X = X_1 \cup X_2$, where X_i is closed in X and $\mathcal{P}$-ind $X_i \leq n \geq 0$ for every $i = 1, 2$. Then $\mathcal{P}$-ind $X \leq n + 1$.*
Moreover, if the space X is normal then for any closed subsets A and B of X there exists a partition C between A and B such that $\mathcal{P}$-ind $C \leq n$.

Corollary 2.1 *Let X be a space and q be an integer. If $X = \bigcup_{k=1}^{n+1} X_k$, where each X_k is closed in X, $0 \leq n \leq 2^m - 1$ for some integer m and $\max\{\mathcal{P}\text{-ind } X_k\} \leq q \geq 0$ then $\mathcal{P}$-ind $X \leq q + m$.*

For every normal space X one assigns *the large inductive compactness degree* Cmp as follows (cf. [1]).

(i) For $n = -1$ or 0, Cmp $X = n$ iff cmp $X = n$.

(ii) Cmp $X \leq n \geq 1$ if each pair of disjoint closed subsets A and B of X there exists a partition C such that Cmp $C \leq n - 1$.

(iii) Cmp $X = n$ if Cmp $X \leq n$ and Cmp $X > n - 1$.

(iv) Cmp $X = \infty$ if Cmp $X > n$ for every natural number n.

It is clear that the following properties of Cmp are valid.

1. If A is closed in X then Cmp $A \leq$ Cmp X.
2. If X is a sum of closed subsets $X_i, i = 1, 2$, then Cmp $X = \max\{\text{Cmp } X_1, \text{Cmp } X_2\}$.

Corollary 2.2 *Let X be a normal space such that $X = X_1 \cup X_2$, where X_i is closed in X and Cmp $X_i \leq 0$ for every i. Then Cmp $X \leq 1$. Moreover, if Cmp $(X_1 \cap X_2) = -1$ then Cmp $X \leq 0$; if Cmp $X_1 = -1$ then Cmp $X =$ Cmp X_2.*

Now we are ready to prove the following theorem.

Theorem 2.2 *Let X be a normal space such that $X = X_1 \cup X_2$, where X_i is closed for $i = 1, 2$. Then Cmp $X \leq \max\{\text{Cmp } X_1, \text{Cmp } X_2\} + \text{Cmp } (X_1 \cap X_2) + 1 \leq \text{Cmp } X_1 + \text{Cmp } X_2 + 1$.*

Proof. Put Cmp $(X_1 \cap X_2) = k$ and $\max\{\text{Cmp } X_1, \text{Cmp } X_2\} = m$. Observe that $k \leq m$. Let $k = -1$. First we will prove the theorem for any $m \geq -1$ $(k = -1)$. By Corollary 2.2 the statement is valid for $m = -1$ and $m = 0$. Assume that our theorem is valid for $m < p \geq 1$. Put $m = p$. Consider two disjoint closed subsets A and B of X. We can suppose that $A \cap X_i \neq \emptyset$ and $B \cap X_i \neq \emptyset, i = 1, 2$. Choose partitions $C_i, i = 1, 2$, as we

did in Lemma 2.1 such that $\max\{\text{Cmp } C_1, \text{Cmp } C_2\} \le p-1$. Denote $Y_1 = C_1 \cup C_2$ (recall that C_1 and C_2 are disjoint), $Y_2 = X_1 \cap X_2$ and $Y = Y_1 \cup Y_2$. Observe that Cmp $(Y_1 \cap Y_2) = -1$, Cmp $Y_1 = \max\{\text{Cmp } C_1, \text{Cmp } C_2\} \le p-1$ and $\max\{\text{Cmp } Y_1, \text{Cmp } Y_2\} \le p-1$. By inductive assumption, Cmp $Y \le \max\{\text{Cmp } Y_1, \text{Cmp } Y_2\} + \text{Cmp } (Y_1 \cap Y_2) + 1 \le -1 + (p-1) + 1 = p-1$. By Lemma 2.1 there is a partition C between A and B in X such that $C \subset Y$. Hence, Cmp $X \le p = k+m+1$.
Assume that our theorem is valid for any pair $(k,m) : k < q \ge 0$ and $k \le m$.
Put $k = q$. Consider the case $m = k \ge 0$. If $k = m = 0$ then Cmp $X_i \le 0$ for every $i = 1, 2$, and by Corollary 2.2, Cmp $X \le 1 = k+m+1$. Let $k = m = q \ge 1$. Consider two disjoint closed subsets A and B of X. We can suppose that $A \cap X_i \neq \emptyset$ and $B \cap X_i \neq \emptyset, i = 1, 2$. Choose partitions $C_i, i = 1, 2$, as we did in Lemma 2.1 such that $\max\{\text{Cmp } C_1, \text{Cmp } C_2\} \le q-1$. Denote $Y_1 = C_1 \cup C_2$ (C_1 and C_2 are disjoint), $Y_2 = X_1 \cap X_2$ and $Y = Y_1 \cup Y_2$. Observe that Cmp $Y_1 = \max\{\text{Cmp } C_1, \text{Cmp } C_2\} \le q-1$, Cmp $(Y_1 \cap Y_2) \le \min\{q, q-1\} = q-1 < q$ and $\max\{\text{Cmp } Y_1, \text{Cmp } Y_2\} \le q$. By inductive assumption, Cmp $Y \le \max\{\text{Cmp } Y_1, \text{Cmp } Y_2\} + \text{Cmp } (Y_1 \cap Y_2) + 1 \le q + (q-1) + 1 = 2q$. By Lemma 2.1 there is a partition C between A and B in X such that $C \subset Y$. Hence, Cmp $X \le 2q+1 = k+m+1$.
Assume that our theorem is valid for any $m : k \le m < p \ge 1$ (k=q). Put $m = p$. Consider two disjoint closed subsets A and B of X. We can suppose that $A \cap X_i \neq \emptyset$ and $B \cap X_i \neq \emptyset, i = 1, 2$. Choose partitions $C_i, i = 1, 2$, as we did in Lemma 2.1 such that $\max\{\text{Cmp } C_1, \text{Cmp } C_2\} \le p-1$. Denote $Y_1 = C_1 \cup C_2$ (C_1 and C_2 are disjoint), $Y_2 = X_1 \cap X_2$ and $Y = Y_1 \cup Y_2$. Observe that Cmp $Y_1 = \max\{\text{Cmp } C_1, \text{Cmp } C_2\} \le p-1$, Cmp $(Y_1 \cap Y_2) \le \min\{q, p-1\} = q$ and $\max\{\text{Cmp } Y_1, \text{Cmp } Y_2\} \le p-1$. By inductive assumption, Cmp $Y \le \max\{\text{Cmp } Y_1, \text{Cmp } Y_2\} + \text{Cmp } (Y_1 \cap Y_2) + 1 \le q + (p-1) + 1 = q+p$. By Lemma 2.1 there is a partition C between A and B in X such that $C \subset Y$. Hence, Cmp $X \le q+p+1 = k+m+1$.

Corollary 2.3 *Let X be a normal space with Cmp $X = n \ge 1$. Then*

(a) X cannot be represented as a union of n many closed subsets $P_1, P_2, \ldots, P_n$ with Cmp $P_i \le 0$ for each i.

Furthermore, we suppose now that $X = \cup_{i=1}^{n+1} Z_i$, where each Z_i is closed and Cmp $Z_i \le 0$ for every $i = 1, \ldots, n+1$, then we have

(b) Cmp $(Z_1 \cup \ldots \cup Z_{k+1}) = k$ for any k with $0 \le k \le n$;

(c) Cmp $((Z_1 \cup \ldots \cup Z_{1+i}) \cap (Z_{i+2} \cup \ldots \cup Z_{i+j+2})) = \min\{i, j\}$ for any nonnegative integers i, j such that $i + j + 1 \le n$.

Remark. The estimations from Corollary 2.2 and Theorem 2.2 can not be improved (see Corollary 3.3).

3 Spaces with cmp $\neq$ def (cmp $\neq$ Cmp).

The deficiency def is defined in the following way: For a separable metrizable space X,

$$\text{def } X = \min\{\text{ind } (Y \setminus X) : Y \text{ is a metrizable compactification of } X\}.$$

In this section, the concept of B-special decomposition introduced in [2] essentially works. A decomposition $X = F \cup \bigcup_{i=1}^{\infty} E_i$ of a metric space X into disjoint sets is called *B-special* if E_i is clopen in X and $\lim_{i\to\infty} \delta(E_i) = 0$, where $\delta(A)$ is the diameter of A.

The following proposition is easily obtained by use of [2, Lemma 2.3].

Proposition 3.1 *Let $X = F \cup \bigcup_{i=1}^{\infty} E_i$ be a B-special decomposition of a metric space X and $n \geq 0$ be an integer. If $\max\{\mathcal{P}\text{-ind } F, \mathcal{P}\text{-ind } E_i\} \leq n$ then $\mathcal{P}\text{-ind } X \leq n$.*

Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of real numbers such that $0 < x_{i+1} < x_i \leq 1$ for all i and $\lim_{i\to\infty} x_i = 0$. Put $C^n = (\text{Bd } I^n \times \{0\}) \cup \bigcup_{i=1}^{\infty}(I^n \times [x_{2i}, x_{2i-1}]) \subset I^{n+1}$.

Theorem 3.1 *(a) There are closed subsets $X_1, X_2, ..., X_{n+1}$ of C^n such that $C^n = \bigcup_{k=1}^{n+1} X_k$ and cmp $X_k = 0$ for each $k = 1, 2, ..., n+1$.*

(b) The equalities def C^n = Cmp $C^n = n$ (= CompC^n) hold (see [1] for the definition of Comp).

(c) Let m be an integer such that $0 \leq n \leq 2^m - 1$. Then we have cmp $C^n \leq m$. In particular cmp $C^n <$ Cmp C^n = def C^n for $n \geq 3$.

Proof. (a) For every i choose finite systems $B_k^i, k = 1, ..., n+1$, consisting of disjoint compact subsets of I^n with diameter $< \frac{1}{i}$ such that $I^n = \bigcup_{k=1}^{n+1}(\cup B_k^i)$. We put $X_k = (\text{Bd } I^n \times \{0\}) \cup \bigcup_{i=1}^{\infty}((\cup B_k^i) \times [x_{2i}, x_{2i-1}])$ for every $k = 1, ..., n+1$. Observe that the space X_k admits a B-special decomposition into compact subsets and, by Proposition 3.1, cmp $X_k = 0$ for every $k = 1, ..., n+1$.
(b) It is enough to prove that Comp $C^n \geq n$ i.e. there exist n pairs $(F_1, G_1), ..., (F_n, G_n)$ of disjoint compact subsets of C^n such that for any partitions S_i between F_i and G_i in $X, i = 1, ..., n$, the intersection $S_1 \cap ... \cap S_n$ is not compact. (Recall that for every separable metrizable space W we have Comp $W \leq$ Cmp $W \leq$ def W (cf. [1]) and evidently def $C^n \leq n$.) For example such pairs are $((\{0\} \times I^n) \cap C^n, (\{1\} \times I^n) \cap C^n), ..., ((I^{n-1} \times \{0\} \times [0,1]) \cap C^n, (I^{n-1} \times \{1\} \times [0,1]) \cap C^n)$.
Moreover, for any partition C between $(\{0\} \times I^n) \cap C^n$ and $(\{1\} \times I^n) \cap C^n$ in C^n, Comp $C \geq n-1$.
(c) One can show (c) by applying Corollary 2.1 for cmp and the statement (a).

Now we are ready to show Theorem 1.1.

Proof of Theorem 1.1. Decompose the space Z_n, $n \geq 3$, into the union of two closed subsets Z_n^1 and Z_n^2 (each of them is homeomorph to C^n), where $Z_n^1 = (\text{Bd } I^n \times \{0\}) \cup \bigcup_{i=1}^{\infty}(I^n \times [1/(2i+1), 1/(2i)])$, $Z_n^2 = (\text{Bd } I^n \times \{0\}) \cup \bigcup_{i=1}^{\infty}(I^n \times [1/(2i), 1/(2i-1)])$.

Let m be the integer such that $0 \leq n \leq 2^m - 1$. It follows from Theorem 3.1 (c) that cmp $Z_n^i \leq m$ for $i = 1, 2$. Thus, by Corollary 2.1, we have cmp $Z_n \leq m + 1$.

Corollary 3.1 *(a) For the space C^2 we have cmp $C^2 =$ cmp $(C^2 \times [0,1]) = 2$.*

(b) cmp $C^3 = 2$.

The following question is discussed in [1, Problem 6, page 71].

Question 3.1 *For any k and m with $0 < k < m$, does there exist a separable metrizable space X such that cmp $X = k$ and def $X = m$?*

We shall partially answer the question as follows.

Corollary 3.2 *Let m be an integer and $l(m) = [log_2(m)] + 1$. Then for every k with $m \geq k \geq l(m)$ there exists a separable metrizable space X such that cmp $X = k$ and def $X = m$.*

Let C^n be the space defined above and $X_1, X_2, \ldots, X_{n+1}$ be closed subsets of C^n described in Theorem 3.1. It follows from Theorem 3.1 (a) and Corollary 2.3 that Cmp $(X_1 \cup \ldots \cup X_{k+1}) = k$ for each k with $0 \leq k \leq n$. However, we do not know the value of the deficiency of $X_1 \cup \ldots \cup X_{k+1}$. So we can ask the following.

Question 3.2 *Is it true that def $(X_1 \cup \ldots \cup X_{k+1}) = k$ for $1 \leq k < n$?*

The question might be interesting when we consider a problem posed by Aarts and Nishiura [1, Problem 6, page 71]: Exhibit a separable metrizable space X such that cmp $X <$ Cmp $X <$ def X. If the Question 3.1 would be answered negatively for example for the case of $n = 4$ and $k = 3$, then we have def $(X_1 \cup X_2 \cup X_3 \cup X_4) = 4$. We put $Y = X_1 \cup X_2 \cup X_3 \cup X_4$. Then, by the argument above, we have Cmp $Y = 3$. On the other hand, by Theorem 3.1 (a) and Corollary 2.1, it follows that cmp $Y \leq 2$. Hence cmp $Y <$ Cmp $Y <$ def Y. Even if the Question 3.1 would be answered positively, then one gets an interesting counterpart of Corollary 3.3 (see below) for def .

Now we will obtain a complement to Theorem 2.2 showing the exactness of the theorem's estimations.

Corollary 3.3 *For any integer $n \geq 1$ there exists a compact space $X_n (= C^n)$ with Cmp $X_n = n$ such that for any nonnegative integers p, q with $p + q = n - 1$ there exist its closed subsets $X_n^{(p)}$ and $X_n^{(q)}$ such that $X_n = X_n^{(p)} \cup X_n^{(q)}$, Cmp $X_n^{(p)} = p$, Cmp $X_n^{(q)} = q$ and Cmp $(X_n^{(p)} \cap X_n^{(q)}) = min\ \{p, q\}$.*

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